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Exactly solvable fermionic N-band models

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Abstract. Motivated by the problem of N-coupled Hubbard chains, we investigate a generalization of a recent model containing two species of one-dimensional fermions interacting via a gauge field that depends on the positions of all the particles of the other species. The exact many-body ground state of the model can be easily obtained through a unitary transformation of the model. The correlation functions are Luttinger like—i.e., they decay through power laws with non-integer exponents. Through the interaction-dependent two-particle correlation functions, we identify the relevant perturbations and hence, possible instabilities. Interestingly, for $N > 2$ bands, beyond a critical strength of the interaction, the dominant incipient instability changes.

Exactly solvable models [1–3]§ have always attracted a lot of interest in theoretical physics, because they serve as paradigms for more complicated systems. The fact that these models are usually in one dimension no longer makes them unrealistic, however, since current technological advances have seen the advent of many semi-artificial one-dimensional systems, such as quantum wires, quantum Hall bars, one-dimensional organic metals and one-dimensional spin chains‖. In fact, phenomena such as one-dimensional Luttinger liquids and the Haldane gap in spin chain models have actually been seen experimentally [5]. Besides their role in these systems, exactly solvable models have played a very important role as a reliable test for various approximation methods and for developing qualitative understanding [6].

However, for two or more dimensions, there have been very few exact results. For instance, the large-U Hubbard model has been studied using several approximation schemes [7], none of which have led to completely reliable results. In recent years, there have been attempts to understand two dimensions through the coupling of one-dimensional chains. Both coupled spin chains [8] and coupled Hubbard models [9] with interchain hopping and interchain interactions have been studied using a variety of different schemes such as weak coupling renormalization group techniques and bosonization [10], exact numerical diagonalizations [11], etc. Unfortunately, in the absence of any exact results, the interpretation of the results of these inter-chain coupling studies has remained difficult [12].

In this paper, with the motivation of approaching two-dimensional phenomena through the coupling of one-dimensional chains, we study a generalization of a class of models [13,14] that can be diagonalized by a pseudo-unitary transformation and still exhibit non-trivial Luttinger

§ For a review and further references, see [4, 6].
‖ For a recent introduction to one-dimensional systems, see [4].

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liquid behaviour. Our model has two species of particles with pseudo-spin index $\sigma = \pm$, at the positions $x_{\sigma i}$ and with momenta $p_{\sigma i}$, with a Hamiltonian given by

$$H = \sum_{I}^{N} \sum_{\sigma i} a_I (\Pi_{\sigma i})^{2I}. \tag{1}$$

Here, $\Pi_{\sigma i} = p_{\sigma i} + \sigma A_\sigma(x_{\sigma i})$ is the 'covariant momentum' introduced in [14] and N is a 'band index'. We have chosen to have only even powers of the covariant momentum in the Hamiltonian, although positivity of energy only requires that the largest power of the covariant momentum be even. This maintains the symmetry $x \to -x$ or parity, which simplifies the presentation of the calculations, although the result goes through even when we include odd powers of the momenta. As explained in [14], particles interact via a gauge potential, given for the particle at the position x by $A_\sigma(x) = \sum_j V(x - x_{-\sigma j})$—i.e., the potential for the particles with positive pseudo-spin is due to the presence of the particles with negative pseudo-spin and vice versa. Note that the potential depends on *all* the particles of the opposite sign of pseudo-spin irrespective of the number of bands. (It is also possible to construct models where the potential only depends on the positions of the particles on nearest-neighbour chains.) The potential is chosen to be an even function, vanishes at infinity and explicitly breaks time-reversal invariance, although it is invariant under a combined operation of time reversal and reversal of pseudospin index. We have generalized the model in [14] by including a band index and allowing higher powers of the covariant momentum in the Hamiltonian. Our model reduces to the Schulz–Shastry model for $I = 1$ and $a_1 = 1$. Clearly, the a_I are not dimensionless, and in fact, explicitly contain a scale Λ (except for a_1, which is dimensionless).

As noted by the authors in [14], the same pseudo-unitary transformation that they use to diagonalize their Hamiltonian,

$$\mathrm{e}^{\mathrm{i}S(\{x_{+i}\},\{x_{-i}\})} p_{\sigma i} \mathrm{e}^{-\mathrm{i}S(\{x_{+i}\},\{x_{-i}\})} = p_{\sigma i} - \partial_{x_{\sigma i}} S(\{x_{+i}\}, \{x_{-i}\}) \tag{2}$$

diagonalizes any power of $\Pi_{\sigma i}$, as long as we choose the function S (a function of the $2n$ positions of the particles) to eliminate the interaction in equation (1). Thus, we obtain the transformed Hamiltonian given by

$$\tilde{H} = \mathrm{e}^{\mathrm{i}S(\{x_{+i}\},\{x_{-i}\})} H \mathrm{e}^{-\mathrm{i}S(\{x_{+i}\},\{x_{-i}\})} = \sum_{\sigma i} \sum_{I}^{N} a_I (p_{\sigma i})^{2I} = \sum_{\sigma i} H_{\sigma i} \tag{3}$$

where the interaction pieces have been removed by the transformation. However, the eigenvalues and eigenfunctions are not the same as those for a genuinely non-interacting Hamiltonian because the boundary conditions on the wavefunctions are now different.

For the single-particle Hamiltonian $H_{\sigma i}$ in equation (3), the eigenvalue equation is a $2N$th order differential equation, and depending on the energy chosen, will have at most $2N$ different solutions. The general solution is given by

$$\tilde{\psi} = \sum_{I}^{N} c_I \mathrm{e}^{\mathrm{i}k_I x} + \text{h.c.} \tag{4}$$

where the k_I are known in terms of E and the $N - 1$ constants a_I (we always choose $a_N = 1$ without loss of generality since it only sets the overall scale). However, not all k_I need be integer multiples of $2\pi/L$, where L is the size of the system. For those that are not, the corresponding c_I vanish so as to make the wavefunction periodic in L. Since the a_I are fixed, we may choose only one of the k_I to be independent, say k, which in turn fixes the dispersion to be

$$E(k) = \sum_{I}^{N} a_I (k^2)^{2I}. \tag{5}$$

The Fermi points are the roots of the equation $E(k) = E_F$. We choose an energy E_F where the $2N$ roots $\{-b_I, b_I\}$ are all real and distinct (with $b_1 < b_2 < \cdots < b_N$). The Fermi points $\{-F_I, F_I\}$ are given by $F_I = 2\pi[r_I]/L$ where $b_I = 2\pi r_I/L$ and $[r_i]$ stands for the largest integer below r_I.

Note that the physics described by an N-band model is similar to the physics described by an N-chain model in the following way. For the N-chain model, first the kinetic energy is diagonalized. By putting periodic or open boundary conditions in the direction perpendicular to the chain, N quantized values of k_y, the momentum perpendicular to the chain direction is obtained and these N values are used to label the N bands or the N dispersion relations [10]. The filling of these bands up to the Fermi level defines the set of $2N$ Fermi points $\pm k_F^I$. Thus, an N-chain model is similar to an N-band model. However, the two models differ in that an N-band model only has a single dispersion (equation (5)). It gets its $2N$ Fermi points because the dispersion is not quadratic and has N wells (unlike the usual quadratic dispersion which has one well per band). Hence, the physics described by the N-band model is similar to that of the N-chain model, but not the same. In particular, the interplay between interchain hopping and intrachain interactions is not well represented here, because here we only have the dispersion and interactions within the chain.

We are interested in the solution of the original Hamiltonian in equation (1) and not the transformed Hamiltonian in equation (3). Although the single-particle energies of the two Hamiltonians are the same, their wavefunctions are related by the pseudo-unitary transformation $\psi = \mathrm{e}^{-\mathrm{i}S}\tilde{\psi}$, where S was chosen to cancel the interaction and is of the form

$$S(\{x_{+i}\}, \{x_{-i}\}) = \sum_{i,j} E(x_{+i} - x_{-j}) \qquad \text{where} \quad E(x) = \int_0^x \mathrm{d}x'\, V(x'). \tag{6}$$

We can compute the difference between $S(x_{-i} = L)$ and $S(x_{-i} = 0)$ for any particular negative pseudo-spin coordinate x_{-i} as

$$S(x_{-i} = L) - S(x_{-i} = 0) = \sum_j [E(x_{+j} - L) - E(x_{+j})] \tag{7}$$

$$= n_+^T \int_0^L V(x)\,\mathrm{d}x \equiv n_+^T \delta \tag{8}$$

in terms of a phase shift δ and n_+^T which is the total number of positive pseudo-spin particles. One gets a similar result if we choose the reference particle to be a positive pseudo-spin particle, with the only difference that n_+^T gets replaced by n_-^T and δ by $-\delta$. Hence the quantization condition on the wavenumbers of the particles becomes

$$Lk_{\pm i} \mp n_\mp^T \delta = 2\pi n_{\pm i} \tag{9}$$

where the $n_{\pm i}$ are integer quantum numbers analogous to those used in the non-interacting case. Since, in general, $n_\mp^T \delta \neq$ integral multiple of 2π, the free Hamiltonian and the interacting Hamiltonian are in different Hilbert spaces.

So far, all the arguments used by Schulz and Shastry have gone through for our model as well. The differences begin when we try to construct the many-body ground state and the spectrum of low-energy excitations. For ease of presentation, we will now specialize to the two-band case, explicitly perform the calculations leading to the low-energy effective Hamiltonian, and then generalize to the case of N bands.

For the two-band case, the single-particle dispersion is given by

$$\tilde{H} = \mathrm{e}^{\mathrm{i}S} H \mathrm{e}^{-\mathrm{i}S} = \sum_{i\sigma} a_4 p_{\sigma i}^4 + a_2 p_{\sigma i}^2 \tag{10}$$

where $a_4 = \Lambda^2$ has length dimension two and $a_2 = -1$ is dimensionless. As mentioned before, we restrict the Fermi level to lie within the double well—i.e., we have four distinct Fermi points

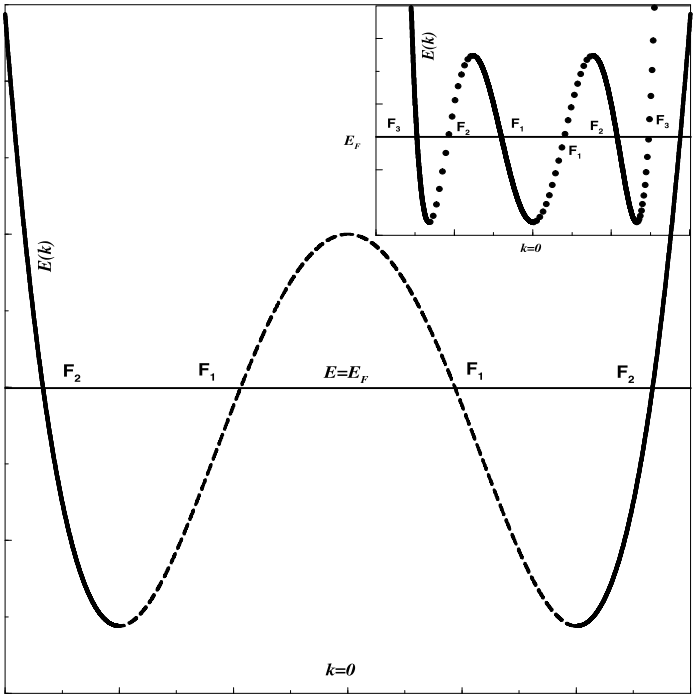

Figure 1. Dispersion for the two-band model. For the Fermi energy E_F, the two bands are denoted by dotted indices (band 1) and full curve (band 2). $(-F_2, F_1)$ are clearly left-mover Fermi points and $(-F_1, F_2)$ are right-mover Fermi points. The inset shows the dispersion of the three-band model, with left-mover branches denoted by full curves and right-mover branches denoted by dotted curves.

$(-F_2, -F_1)$ on the left and (F_1, F_2) on the right. For each energy, the degeneracy is either four or two, depending on whether or not both k_1 and k_2 satisfies the boundary condition given in equation (9). However, all that really matters is that energy levels in both the wells below the Fermi level are filled. Let us assume that in band 1, there are n_1 states below the Fermi level and in band 2, there are n_2 states below the Fermi level for both pseudo-spins. The ground state energy is then given by

$$E_{gs} = 2\sum_{\sigma=\pm}\sum_{i=-n_1}^{n_2}[h_1 i^4 - h_2 i^2] \tag{11}$$

where $h_1 = \Lambda^2(\frac{2\pi}{L})^4$ and $h_2 = (\frac{2\pi}{L})^2$. We have assumed that $n_{\pm}^T\delta/2\pi$ is an integer and the factor of two takes care of the contributions from both the right- and left-moving sectors. (Note that the states on the right branches of both the wells are right movers, whereas the states on the left branches of both wells are left movers. See figure 1.)

The second-order fluctuation in the energy due to the addition of $n_{R\pm I}$ and $n_{L\pm I}$ particles for the right and left movers can also be computed. It is given by

$$E^{(2)} = \left[\sum_{\pm}\sum_{-(n_1-n_{R\pm1})}^{(n_2+n_{R\pm2})}[h_1(i+c_{\mp})^4 - h_2(i+c_{\mp})^2] + R \longrightarrow L\right]_2 \tag{12}$$

$$= \sum_{I}^{2}\sum_{\pm}[h_1 f_1(n_I) + h_2 f_2(n_I)](n_{R\pm I} - c_{\mp})^2 \tag{13}$$

$$= \sum_{\pm} \sum_{I}^{2} g(n_I)[J_{\pm I} \pm n_{\mp}\delta/2\pi)^2 + (n_{\pm I})^2] \tag{14}$$

where $c_\sigma = n_\sigma^T \delta/2\pi$ in the first row and the subscript 2 is to indicate that we keep only terms up to quadratic order in the fluctuations. In the second row, $f_1(n_I) = n_I(n_I + 1)(2n_I + 1)$ and $f_2 = n_I + \frac{1}{2}$, and in the third row, we have defined the current $J_{\pm I} = n_{R\pm I} - n_{L\pm I}$, the charge $n_{\pm I} = n_{R\pm I} + n_{L\pm I}$ and the 'density' $g(n_I) = h_1 f_1(n_I) + h_2 f_2(n_I)$. The total charge is clearly $n_\pm^T = n_{\pm 1} + n_{\pm 2}$. Note, however, that unlike the Schulz–Shastry model, here fourth-order fluctuations do exist, which we neglect because we are only interested in low-energy fluctuations.

We bosonize as in the single-band case by introducing boson fields $\phi_{\pm I}$ with their conjugate momenta $\Pi_{\pm I}$. These are related to the currents and charge densities as

$$n_{\pm I} = \frac{L}{\sqrt{\pi}} \partial_x \phi_{\pm I} \tag{15}$$

$$J_{\pm I} = -\frac{L}{\sqrt{\pi}} \Pi_{\pm I}. \tag{16}$$

(We use the notation of the first reference in [15].) To rewrite the effective Hamiltonian for the low-energy fluctuations in terms of the boson fields, we have to identify the function $g(n_I)L/2\pi = \rho_I$ as an effective density after which we obtain

$$H = \sum_{I}^{2} \sum_{\pm} \int \mathrm{d}x\, \rho_I \left\{ \left[-\Pi_{I\pm} \pm \frac{\delta}{\pi}(\partial_x \phi_{1\mp} + \partial_x \phi_{2\mp}) \right]^2 + (\partial_x \phi_{\pm I})^2 \right\}. \tag{17}$$

But interestingly, although ρ_I contains information about the scale, the low energy effective Hamiltonian is scale invariant—there are no mass terms (or cosine terms leading to mass terms) for the boson fields. A similar redefinition of variables as in the one-band case,

$$\tilde{\phi}_{\pm I} = \phi_{\pm I} \qquad \tilde{\Pi}_{\pm I} = \Pi_{\pm I} \mp \frac{\delta}{\pi}(\partial_x \phi_{\mp 1} + \partial_x \phi_{\mp 2}) \tag{18}$$

leads to a non-interacting form of the Hamiltonian given by

$$H = \sum_{I}^{2} \sum_{\pm} \int \mathrm{d}x\, \rho_I [(\tilde{\Pi}_{\pm I})^2 + (\partial_x \tilde{\phi}_{\pm I})^2]. \tag{19}$$

Thus the correlators of the tilde fields are just free-field correlators. In terms of the non-tilde bosonic variables or equivalently in terms of the fermion fields, the Hamiltonian is not non-interacting. However, since they are explicitly known in terms of the free fields, their correlators can also be explicitly calculated.

In fact, at this stage, the generalization to N bands is obvious. The single-particle dispersion of the N-band model has N wells and $2N$ Fermi points. The Hamiltonian for quadratic fluctuations about the Fermi points is precisely the same as that in equation (17) with the replacement

$$(\partial_x \phi_{\mp 1} + \partial_x \phi_{\mp 2}) \to \sum_{J}^{N} \partial_x \phi_{\mp J}. \tag{20}$$

As before, the redefinition of ϕ_I and Π_I in terms of the tilde fields leads to the non-interacting form of the Hamiltonian in equation (19) with the sum going over all N bands.

We can now compute correlation functions using the representation for the fermion operators in terms of the non-interacting boson fields given by

$$\psi_{R\pm I} = \exp\left\{ \mathrm{i} \left(\phi_{R\pm I} \mp \frac{\delta}{2\pi} \sum_{I}^{N} \phi_{\mp I} \right) \right\} \tag{21}$$

Table 1. Two-particle correlations.

TPO	η	TPO($I \neq J$)	η_2
$\psi^\dagger_{R\pm I}\psi_{L\pm J}$	2	$\psi^\dagger_{Z\pm I}\psi_{Z\pm J}$	$2\left(1+N\frac{\delta^2}{\pi^2}\right)$
$\psi^\dagger_{R\pm I}\psi_{L\mp J}$	$2\left(1\mp\frac{\delta}{\pi}\right)+N\frac{\delta^2}{\pi^2}$	$\psi^\dagger_{Z\pm I}\psi_{Z\mp J}$	$2\left(1+\frac{N\delta^2}{2\pi^2}\right)$
$\psi_{R\pm I}\psi_{L\pm J}$	$2\left(1+N\frac{\delta^2}{\pi^2}\right)$	$\psi_{Z\pm I}\psi_{Z\pm J}$	2
$\psi_{R\pm I}\psi_{L\mp J}$	$2\left(1\pm\frac{\delta}{\pi}\right)+N\frac{\delta^2}{\pi^2}$	$\psi_{Z\pm I}\psi_{Z\mp J}$	$2\left(1+\frac{N\delta^2}{2\pi^2}\right)$

$$\psi_{L\pm I}=\exp\left\{-\mathrm{i}\left(\phi_{L\pm I}\pm\frac{\delta}{2\pi}\sum_I^N\phi_{\mp I}\right)\right\}. \tag{22}$$

(We follow the notation in [15] and define $\phi_{R\pm I}=\frac{1}{2}(\phi_{\pm I}-\int_{-\infty}^{x}\Pi_{\pm I}(x')\,\mathrm{d}x')$ and $\phi_{L\pm I}=\frac{1}{2}(\phi_{\pm I}+\int_{-\infty}^{x}\Pi_{\pm I}(x')\,\mathrm{d}x')$.) The one-particle correlation function is given by

$$G_{Z\pm I}(x)=\langle\psi_{Z\pm I}(x)\psi^\dagger_{Z\pm I}(0)\rangle\sim x^{-\eta} \tag{23}$$

with $\eta=1+N(\delta^2/2\pi^2)$ for both right and left movers ($Z=R/L$), for both pseudo-spins and for all I. As in the one-band case, the fermion has an anomalous dimension given by $\eta\neq$ integer. This is the indication that the system is a Luttinger liquid and not a Fermi liquid. The interesting point to note here is the dependence of the anomalous dimension on the number of chains. The model is not just a collection of one-band Luttinger liquids—there exists a genuine dependence on the number of bands.

To study the incipient instabilities of the model, we need to compute the anomalous dimensions of multi-particle operators. Naive power counting using the engineering dimension of the fermion field as $\frac{1}{2}$, gives the engineering dimension of any M-particle operator correlation function $\langle O_M O_M^\dagger\rangle$ as $2M/2=M$. Hence, without interactions, one would expect two-particle operators (TPOs) to be marginal and all higher-particle operators to be irrelevant. However, with interactions, it is clearly possible for some TPOs or even higher-particle operators to become relevant. Hence, to study incipient instabilities, we need to compute the anomalous dimensions of all possible operators. For N-chains, one can include up to $4N$-particle operators (because each particle can be left or right moving and can be of the + or − type) without violating the Pauli exclusion principle. However, generally the lower-point functions tend to be more relevant and hence lead to more likely ground states.

Hence, let us first compute exponents of the TPOs. In the one-band case, the only non-trivial two-particle correlations involved excitations at both the right and left Fermi points, because these were the only two Fermi points. Here, however, we can have non-trivial two-particle correlations involving excitations at two right Fermi points and two left Fermi points as well. These exponents for the two-particle correlations are tabulated in table 1.

Fortunately, they are independent of the band index and only depend on whether they involve both right and left Fermi points or right (left) movers at both Fermi points. Interestingly, none of the RR or LL exponents lead to relevant perturbations. This is in agreement with the weak coupling RG approach [16], where there is a non-zero contribution to the four-point vertex only when there is momentum transfer between left and right Fermi points. In the single-chain case considered in [16], there was no possibility of momentum transfers between two Fermi points on the left or two on the right, since the model only had one on each side. However, even in the more general case of N left-moving Fermi points and N right-moving Fermi points [17], graphs involving loop momenta in two left-moving shells or two right-moving shells are always zero, because the energies have the same sign and the contour integral for the energy vanishes.

For positive δ, the favoured relevant perturbations are

$$\begin{array}{ll} \psi^{\dagger}_{R+I}\psi_{L-J} & \eta = 1 + (1-\delta/\pi)^2 + (N-1)\delta^2/\pi^2 \\ \psi_{R-I}\psi_{L+J} & \eta = 1 + (1-\delta/\pi)^2 + (N-1)\delta^2/\pi^2. \end{array} \tag{24}$$

Clearly as the number of chains increases, the exponent increases, until at some critical value of $N = N_c \propto 1/\delta$, both the operators above cease to be relevant. Similarly, for negative δ, the most favoured relevant perturbations are

$$\begin{array}{ll} \psi^{\dagger}_{R-I}\psi_{L+J} & \eta = 1 + (1+\delta/\pi)^2 + (N-1)\delta^2/\pi^2 \\ \psi_{R+I}\psi_{L-J} & \eta = 1 + (1+\delta/\pi)^2 + (N-1)\delta^2/\pi^2 \end{array} \tag{25}$$

which again cease to be relevant beyond N_c. Note also that even before the operator ceases to be relevant, it ceases to be the most favoured relevant operator. Beyond $\delta_c/\pi = 2/N$, the charged density wave-like operator $\psi^{\dagger}_{R\pm I}\psi_{L\pm J}$ becomes the most relevant operator.

What about higher-particle operators? Are they always less relevant than the TPOs? To check that, we computed the anomalous dimensions of the various three- and four-particle operators and checked that they never become more relevant than *all* the TPOs for any δ/π between zero and one. Hence, the dominant incipient instability is still one of the ground states determined by a two-particle perturbation. The instabilities characterized by the various two-body operators have been studied before and will lead to the usual charged density wave, spin density wave, triplet superconductor or singlet superconductor states.

Let us compare our results with the results obtained by giving additional internal degrees of freedom to the $\sigma = \pm$ particles [14]. If we assume that they occur in m flavours, then the Hamiltonian is just

$$H = \sum_{I=1}^{m}\sum_{\sigma i} a_I(\Pi_{I\sigma i})^2. \tag{26}$$

Surprisingly, an analogous calculation leads precisely to the same exponents as in equations (24) and (25) with N replaced by m. However, in this case, there genuinely exist $2m$ degrees of freedom, and the various two-particle correlators have physical meaning. $I = J$ give two-particle correlators of the Ith particle, whereas $I \neq J$ give correlations between two different flavours. The exponents are actually independent of the particle index because of the internal symmetry. For our Hamiltonian in equation (1), however, there are only two degrees of freedom correponding to the $\sigma = \pm$ particles. It is only after linearizing around the different Fermi points and assuming that each of the linearized fermions can be bosonized independently that we have N independent right- and left-moving fermions or bosons, whose correlators can be computed independently. For the original fermions, the only relevant charges are $n^T_{\pm}$ and the relevant currents are $J_{\pm} = \sum_I (n_{R\pm I} - n_{L\pm I})$.

Many of the issues in coupled chain models, however, remain unaddressed in this rather simple model, which is perhaps better thought of as a single-chain model with a more complicated band structure. To really apply this model to N-chains, one would have to modify the model, so that there is some analogue of the interplay between interchain hopping and intrachain interactions. However, note that even as an N-band model, it is not trivial that the correlation functions are identical to those of the m-flavour model.

In conclusion, we have studied a general model with $2N$ Fermi points, (an N-band model), which is exactly solvable and has non-trivial Luttinger liquid behaviour. We computed the exponents of the various TPOs and found the possible relevant perturbations. Interestingly, we found that the exponents have non-trivial dependence on the number of bands—they are not merely additive. For $N > 2$, we found that for sufficiently large $\delta > \delta_c = 2\pi/N$, the dominant incipient instability changes to being CDW like from being SDW like at weak δ.

An interesting exercise would be to see if this model can be effectively generalized to higher dimensions. In that case, it would be a good starting point to study possible Luttinger liquid ground states in higher dimensions.

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